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Comments on the global constraints in light-cone string and membrane theories

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Abstract: In the light-cone closed string and toroidal membrane theories, we associate the global constraints with gauge symmetries. In the closed string case, we show that the physical states defined by the BRS charge satisfy the level-matching condition. In the toroidal membrane case, we show that the Faddeev-Popov ghost and anti-ghost corresponding to the global constraints are essentially free even if we adopt any gauge fixing condition for the local constraint. We discuss the quantum double-dimensional reduction of the wrapped supermembrane with the global constraints.

Keywords: Bosonic Strings, M-Theory, BRST Symmetry.

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Contents

1. Introduction

Light-cone gauge formalism is useful in string/M-theory.[1] In M-theory, we know the light-cone gauge formalism [1], while the covariant formalism is not yet known. In the light-cone gauge, string/M-theory is formulated with only the transverse degrees of freedom, however, all of them are not independent but are subject to the constraints. For example, there exists the level-matching condition in the closed string theory. Such constraints originate from the residual symmetries in the light-cone gauge: The residual symmetries are the length-preserving and the area-preserving diffeomorphisms in the closed string and membrane theory, respectively.

On the membrane with the non-trivial space-sheet topology, we can decompose the generators into the co-exact and harmonic parts, which correspond to the local and global constraints for the transverse degrees of freedom, respectively [2, 3, 4].[2] The co-exact part generates an invariant subgroup of the group of the area-preserving diffeomorphisms [4]. The subgroup can be viewed as the $N \to \infty$ limit of $\mathrm{U}(N)$ and hence it can be regularized by a finite dimensional group $\mathrm{U}(N)$ [5, 6].

It is well known that the supermembrane in eleven dimensions is related to type-IIA superstring in ten dimensions through the double-dimensional reduction. It was shown classically [7], however, it is not obvious whether it holds also in quantum theory [8, 9]. Recently, several authors [9, 10] analyzed the double-dimensional reduction quantum mechanically with the light-cone wrapped supermembrane action. In the analyses, however, the global constraints were not taken into account, though the space-sheet topology of the wrapped supermembrane is a torus.

The purpose of this paper is to formulate the light-cone toroidal membrane theory as a gauge theory of the area-preserving diffeomorphisms incorporating the global constraints. And based on the result, we reconsider the problem of the quantum mechanical double-dimensional reduction with the light-cone wrapped supermembrane action. The plan of

[1]In this paper, we use a convention of the light-cone coordinates as $x^\mu = (x^+, x^-, x^i)$ where $x^\pm = (x^0 \pm x^{D-1})/\sqrt{2}$ and the transverse coordinates are x^i $(i = 1, 2, \ldots, (D-2))$.

[2]In the length-preserving diffeomorphisms of the closed string, there is no local constraint, though the global constraint exists.

this paper is as follows. In section 2, as a warm-up, we formulate light-cone closed string theory as a gauge theory of the length-preserving diffeomorphisms. We fix the gauge and calculate the BRS charge to determine the physical states. In section 3, we formulate light-cone toroidal membrane theory as a gauge theory of (the total group of) the area-preserving diffeomorphisms. In this case also, we fix the gauge and calculate the BRS charge. By using the gauge fixed action, we discuss the quantum double-dimensional reduction of the wrapped supermembrane with the global constraints.

2. Gauging the light-cone closed string

Our starting point is the following Polyakov action of a (bosonic) closed string,

$$S_0^{st} = -\frac{T_{st}}{2}\int d\tau d\sigma\, \sqrt{-g}\, g^{ab}\partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu}\,, \tag{2.1}$$

where the indices μ, ν run through 0,1,. . .,25 and a, b take τ, σ. Adopting the conformal gauge of the world-sheet metric, we obtain the following action and constrains,

$$S_{cg}^{st} = \frac{T_{st}}{2}\int d\tau d\sigma\, \left[(\partial_\tau X^\mu)^2 - (\partial_\sigma X^\mu)^2\right], \tag{2.2}$$

$$(\partial_\tau X^\mu)^2 + (\partial_\sigma X^\mu)^2 = 0\,, \tag{2.3}$$

$$\partial_\tau X_\mu \partial_\sigma X^\mu = 0\,, \tag{2.4}$$

and the canonical momentum is given by $P^\mu = T_{st}\partial_\tau X^\mu$. As is well known, there is a residual gauge symmetry in the action (2.2). To fix the residual gauge symmetry, we adopt the light-cone gauge,

$$X^+ = x^+ + \frac{p^+}{2\pi T_{st}}\,\tau\,, \tag{2.5}$$

where p^+ is the total light-cone momentum defined by $p^+ = \int_0^{2\pi} d\sigma P^+$. In the light-cone gauge, the action (2.2) and the constraints (2.3) and (2.4) are rewritten by

$$S_{lc}^{st} = \frac{T_{st}}{2}\int d\tau \int_0^{2\pi} d\sigma\, \left[(\partial_\tau X^i)^2 - (\partial_\sigma X^i)^2\right], \tag{2.6}$$

$$\partial_\tau X^- = \frac{\pi T_{st}}{p^+}\left[(\partial_\tau X^i)^2 + (\partial_\sigma X^i)^2\right], \tag{2.7}$$

$$\partial_\sigma X^- = \frac{2\pi T_{st}}{p^+}\partial_\tau X^i \partial_\sigma X^i\,. \tag{2.8}$$

Eq. (2.7) stands for the hamiltonian $P^- = T_{st}\partial_\tau X^-$ which is derived from the action (2.6). Eq. (2.8) can be solved for X^- in terms of the transverse coordinates X^i (with an undetermined integration constant). In the light-cone gauge, we can describe the system only by the transverse coordinates X^i, however, all the transverse degrees of freedom are not always independent. Actually, by integrating over σ on both sides of eq. (2.8), we obtain the following global constraint,

$$\int_0^{2\pi} d\sigma(\partial_\tau X^i \partial_\sigma X^i) = 0\,. \tag{2.9}$$

Hence the light-cone closed string theory is described by the action (2.6) with the global constraint (2.9).

Now we incorporate the global constraint (2.9) as a gauge symmetry. First, note that the action (2.6) is invariant under a constant shift of σ coordinate,

$$\delta X^i = -\epsilon_0\, \partial_\sigma X^i\,, \tag{2.10}$$

where the parameter ϵ_0 is independent of both τ and σ coordinates, and hence this is a global transformation. Next, we extend the global invariance of the action (2.6) into a local one with respect to τ coordinate by introducing a gauge degree of freedom. We have the following gauge theory,

$$\begin{aligned} S_g^{st} &= \frac{T_{st}}{2}\int d\tau \int_0^{2\pi} d\sigma\, \left[(D_\tau X^i)^2 - (\partial_\sigma X^i)^2\right]\,, \\ D_\tau X^i &= \partial_\tau X^i + u\, \partial_\sigma X^i\,, \end{aligned} \tag{2.11}$$

where the gauge degree of freedom $u = u(\tau)$ depends only on τ coordinate and is independent of σ coordinate. The gauge theory (2.11) is invariant under the gauge transformations,

$$\delta X^i = -\epsilon\, \partial_\sigma X^i\,, \tag{2.12}$$

$$\delta u = \partial_\tau \epsilon\,. \tag{2.13}$$

Note that the parameter $\epsilon = \epsilon(\tau)$ also depends only on τ coordinate and is independent of σ coordinate. The transformations generate (time-dependent) length-preserving diffeomorphisms. It is easy to show the equivalence between the gauge theory (2.11) and the mechanical system of the action (2.6) with the global constraint (2.9): In fact, the Euler-Lagrange equations for X^i and u derived from S_g^{st} (2.11) are given by

$$D_\tau^2 X^i - \partial_\sigma^2 X^i = 0\,, \tag{2.14}$$

$$\int_0^{2\pi} d\sigma (D_\tau X^i \partial_\sigma X^i) = 0\,. \tag{2.15}$$

If we choose $u = 0$ as a gauge fixing condition, eqs. (2.14) and (2.15) agree with the Euler-Lagrange equation for X^i derived from the action (2.6) and the global constraint (2.9), respectively. Thus we have shown the equivalence between the gauge theory (2.11) and the mechanical system of the action (2.6) with the global constraint (2.9), i.e., the light-cone closed string theory.

We should notice that the above equivalence is classical. Our next task is to discuss the quantum-mechanical equivalence. We adopt $u + \xi B/2$ (ξ : gauge parameter) as a gauge fixing function. Then, according to a standard procedure of gauge fixing [11], the gauge fixed action of the gauge theory (2.11) is given by

$$\tilde{S}_g^{st} = \int d\tau \int_0^{2\pi} d\sigma \left[\frac{1}{2}(D_\tau X^i)^2 - \frac{1}{2}(\partial_\sigma X^i)^2 + i\bar{c}\,\partial_\tau c + Bu + \frac{\xi}{2}B^2\right]\,, \tag{2.16}$$

where the Faddeev-Popov (FP) ghost $c = c(\tau)$, anti-ghost $\bar{c} = \bar{c}(\tau)$ and Nakanishi-Lautrap (NL) B-field[3] $B = B(\tau)$ depend only on τ coordinate and they are independent of σ

[3]Precisely speaking, it should be refereed to as NL B-degree of freedom since it is not a field.

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coordinate. In the above gauge fixed action, we have set $T_{st} = 1$ for brevity. The action (2.16) is invariant under the BRS transformations,

$$\begin{aligned} \delta_B X^i &= -c\,\partial_\sigma X^i\,, \\ \delta_B u &= \partial_\tau c\,, \\ \delta_B c &= 0\,, \\ \delta_B \bar{c} &= iB\,, \\ \delta_B B &= 0\,, \end{aligned} \tag{2.17}$$

and the BRS charge is given by

$$Q_B = -\int_0^{2\pi} d\sigma (D_\tau X^i \partial_\sigma X^i)\, c\,. \tag{2.18}$$

By using Dirac's method for constraint systems, it is straightforward to quantize the action (2.16) in the operator formalism. Actually, we obtain the following canonical (anti-)commutation relations,

$$[\,X^i(\tau,\sigma), P^j(\tau,\sigma')\,] = i\delta^{ij}\,\delta(\sigma-\sigma')\,, \tag{2.19}$$

$$[\,u(\tau), X^i(\tau,\sigma)\,] = -i\frac{\xi}{2\pi}\partial_\sigma X^i(\tau,\sigma)\,, \tag{2.20}$$

$$[\,u(\tau), P^i(\tau,\sigma)\,] = -i\frac{\xi}{2\pi}\partial_\sigma P^i(\tau,\sigma)\,, \tag{2.21}$$

$$[\,B(\tau), X^i(\tau,\sigma)\,] = \frac{i}{2\pi}\partial_\sigma X^i(\tau,\sigma)\,, \tag{2.22}$$

$$[\,B(\tau), P^i(\tau,\sigma)\,] = \frac{i}{2\pi}\partial_\sigma P^i(\tau,\sigma)\,, \tag{2.23}$$

$$\{\,c(\tau), \bar{c}(\tau)\,\} = \frac{1}{2\pi}\,, \tag{2.24}$$

where $P^i = D_\tau X^i$ is the canonical momentum of X^i. Furthermore, we give the Euler-Lagrange equations,

$$\frac{\delta \tilde{S}_g^{st}}{\delta X^i} = 0 \Rightarrow D_\tau^2 X^i - \partial_\sigma^2 X^i = 0\,, \tag{2.25}$$

$$\frac{\delta \tilde{S}_g^{st}}{\delta u} = 0 \Rightarrow \frac{1}{2\pi}\int_0^{2\pi} d\sigma (D_\tau X^i \partial_\sigma X^i) + B = 0\,, \tag{2.26}$$

$$\frac{\delta \tilde{S}_g^{st}}{\delta B} = 0 \Rightarrow u + \xi B = 0\,, \tag{2.27}$$

$$\frac{\delta \tilde{S}_g^{st}}{\delta c} = \frac{\delta \tilde{S}_g^{st}}{\delta \bar{c}} = 0 \Rightarrow \partial_\tau \bar{c} = \partial_\tau c = 0\,. \tag{2.28}$$

The above Euler-Lagrange equations are, of course, consistent with the Heisenberg equations for the action (2.16) obtained by using the canonical commutation relations (2.19)–(2.24) and the hamiltonian.

Next, we solve the Euler-Lagrange equations (2.25)–(2.28). Henceforth, we choose the Landau gauge ($\xi = 0$) only for simplicity. In the Landau gauge, we have $u = 0$ from

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eq. (2.27), and hence $D_\tau = \partial_\tau$. Then we can solve eq. (2.25) in terms of the oscillation modes,

$$X^i(\tau,\sigma) = x^i + \frac{p^i}{2\pi}\,\tau + \frac{i}{2\sqrt{\pi}}\sum_{n\neq 0}\left(\frac{1}{n}\alpha_n^i\, e^{-in(\tau-\sigma)} + \frac{1}{n}\tilde{\alpha}_n^i\, e^{-in(\tau+\sigma)}\right), \tag{2.29}$$

where use has been made of the boundary condition of the closed string, $X^i(\tau,\sigma) = X^i(\tau,\sigma+2\pi)$. The canonical momentum P^i is given by

$$P^i(\tau,\sigma) = \partial_\tau X^i(\tau,\sigma) = \frac{p^i}{2\pi} + \frac{1}{2\sqrt{\pi}}\sum_{n\neq 0}\left(\alpha_n^i\, e^{-in(\tau-\sigma)} + \tilde{\alpha}_n^i\, e^{-in(\tau+\sigma)}\right). \tag{2.30}$$

Eq. (2.19) and the above solutions lead to the commutation relations for the oscillation modes,

$$[x^i, p^j] = i\delta^{ij}\,, \qquad [\alpha_n^i, \alpha_m^j] = [\tilde{\alpha}_n^i, \tilde{\alpha}_m^j] = n\,\delta_{n+m,0}\,\delta^{ij}\,. \tag{2.31}$$

Then we can construct the Fock space for the X^i sector. Furthermore, from eqs. (2.26) and (2.29), we can solve for B in terms of the oscillation modes,

$$B(\tau) = -\frac{1}{2\pi}\int_0^{2\pi} d\sigma\,\partial_\tau X^i \partial_\sigma X^i = \frac{1}{2\pi}(N - \tilde{N})\,, \tag{2.32}$$

where $N = \sum_{n>0}\alpha_{-n}^i\alpha_n^i$ and $\tilde{N} = \sum_{n>0}\tilde{\alpha}_{-n}^i\tilde{\alpha}_n^i$. As for the FP ghost sector, from eqs.(2.28) we can solve for c and $\bar{c}$,

$$c(\tau) = c_0\,, \qquad \bar{c}(\tau) = \bar{c}_0. \tag{2.33}$$

Since the canonical anti-commutation relation is given by

$$\{c_0, \bar{c}_0\} = \frac{1}{2\pi}\,, \tag{2.34}$$

the Fock space for the FP ghost sector is spanned by two states, $|\!\uparrow\rangle$ and $|\!\downarrow\rangle$, which satisfy

$$c_0|\!\downarrow\rangle = |\!\uparrow\rangle\,, \qquad \bar{c}_0|\!\uparrow\rangle = |\!\downarrow\rangle\,. \tag{2.35}$$

Finally, we define the physical states according to the standard procedure [12]. Substituting eqs. (2.29) and (2.33) into the BRS charge (2.18), we obtain

$$Q_B = (N - \tilde{N})\,c_0\,. \tag{2.36}$$

We adopt the state $|\!\downarrow\rangle$ as a physical state of the FP ghost sector.[4] Then the physical state condition,

$$Q_B|phys\rangle \otimes |\!\downarrow\rangle = (N - \tilde{N})|phys\rangle \otimes |\!\uparrow\rangle = 0\,, \tag{2.37}$$

leads to the condition that the physical states of the X^i sector ($|phys\rangle$) satisfy the level-matching condition ($N - \tilde{N} = 0$). Thus, we have shown that the physical states in the gauge theory (2.11) agree with the states in the light-cone closed string theory of the action (2.6) with the global constraint (2.9), i.e., both theories are equivalent quantum mechanically.

[4] A similar degeneracy of the two zero-mode states in the FP ghost sector appears in the covariant quantization of a bosonic string based on the BRS invariance [13].

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3. Gauging the light-cone toroidal membrane

In this section we start off with the following Polyakov-type action of a (bosonic) membrane,

$$S_0 = -\frac{T}{2}\int d\tau d\sigma d\rho \sqrt{-g}\,(g^{ab}\partial_a X^\mu \partial_b X^\nu\,\eta_{\mu\nu} - 1)\,, \tag{3.1}$$

where the indices μ, ν run through $0, 1, \ldots, D-1$ and a, b take τ, σ and ρ. We adopt the following gauge for the world-volume metric,

$$g_{ab} = \begin{pmatrix} -L^{-2}h & \mathbf{0} \\ \mathbf{0} & h_{\alpha\beta} \end{pmatrix}, \tag{3.2}$$

where $h_{\alpha\beta}$ $(\alpha, \beta = \sigma, \rho)$ is a metric on the space-sheet of the membrane, $h = \det h_{\alpha\beta}$ and L is an arbitrary length parameter.[5] In this gauge the action and the constraints are given by

$$S_{cg} = \frac{LT}{2}\int d\tau d\sigma d\rho \left[(\partial_\tau X^\mu)^2 - \frac{1}{2L^2}\{X^\mu, X^\nu\}^2\right], \tag{3.3}$$

$$(\partial_\tau X^\mu)^2 + \frac{1}{2L^2}\{X^\mu, X^\nu\}^2 = 0\,, \tag{3.4}$$

$$\partial_\tau X_\mu\, \partial_\alpha X^\mu = 0\,, \tag{3.5}$$

where the Poisson bracket is defined by

$$\{X^\mu, X^\nu\} \equiv \epsilon^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu = \partial_\sigma X^\mu \partial_\rho X^\nu - \partial_\rho X^\mu \partial_\sigma X^\nu\,. \tag{3.6}$$

The canonical momentum is given by $P^\mu = LT\partial_\tau X^\mu$. In this case also, there is a residual symmetry in S_{cg} (3.3) and in order to fix it we adopt the light-cone gauge,

$$X^+ = x^+ + \frac{p^+}{(2\pi)^2 LT}\tau\,, \tag{3.7}$$

where p^+ is the total light-cone momentum defined by $p^+ = \int_0^{2\pi} d\sigma d\rho\, P^+$. In the light-cone gauge, the action (3.3) and the constraints (3.4) and (3.5), respectively, are rewritten by

$$S_{lc} = \frac{LT}{2}\int d\tau \int_0^{2\pi} d\sigma d\rho \left[(\partial_\tau X^i)^2 - \frac{1}{2L^2}\{X^i, X^j\}^2\right], \tag{3.8}$$

$$\partial_\tau X^- = \frac{(2\pi)^2 LT}{2p^+}\left[(\partial_\tau X^i)^2 + \frac{1}{2L^2}\{X^i, X^j\}^2\right], \tag{3.9}$$

$$\partial_\alpha X^- = \frac{(2\pi)^2 LT}{p^+}\partial_\tau X^i \partial_\alpha X^i\,. \tag{3.10}$$

In this gauge eq. (3.9) stands for the hamiltonian, $P^- = LT\partial_\tau X^-$, for the action (3.8). Eq. (3.10) can be solved for X^- in terms of the transverse coordinates X^i (with an undetermined integration constant). Thus, in the light-cone gauge, the system is described only

[5] Note that the mass dimensions of the parameters, τ, σ and ρ, are 0 and that of the world-volume metric g_{ab} is -2.

with the transverse coordinates X^i. However, all the transverse degrees of freedom are not always independent. The integrability condition for eq. (3.10), $\epsilon^{\alpha\beta}\partial_\alpha\partial_\beta X^- = 0$, leads to

$$\Phi_0(\sigma,\rho) \equiv \{\partial_\tau X^i, X^i\} = 0\,. \tag{3.11}$$

This is locally equivalent to eq. (3.10) and we call it the local constraint.[6] As for the membrane with the non-trivial space-sheet topology, we must further impose the global constraints. Henceforth we consider the toroidal membrane for definiteness. Then, X^- is a periodic function with respect to both σ and ρ coordinates. Integrating both sides of eq. (3.10) over either σ or ρ for $\alpha=\sigma$ or $\alpha=\rho$, respectively, we obtain the following global constraints,

$$\Phi_1(\rho) \equiv \int_0^{2\pi} d\sigma(\partial_\tau X^i \partial_\sigma X^i) = 0\,, \tag{3.12}$$

$$\Phi_2(\sigma) \equiv \int_0^{2\pi} d\rho(\partial_\tau X^i \partial_\rho X^i) = 0\,. \tag{3.13}$$

Such global constraints (3.12) and (3.13), however, are not completely independent of the local constraint (3.11). In fact, by integrating (3.11) over σ, we get

$$0 = -\int_0^{2\pi} d\sigma\, \Phi_0(\sigma,\rho) = \partial_\rho \int_0^{2\pi} d\sigma(\partial_\tau X^i \partial_\sigma X^i) = \partial_\rho\, \Phi_1(\rho)\,. \tag{3.14}$$

This means that the constraint (3.12) is already included in the local constraint (3.11) except for the ρ-independent mode of $\Phi_1(\rho)$. Thus, in addition to the local constraint (3.11) we may impose the following global constraint instead of eq. (3.12),

$$\Phi_\sigma \equiv \int_0^{2\pi} d\sigma d\rho(\partial_\tau X^i \partial_\sigma X^i) = 0\,. \tag{3.15}$$

Similarly, instead of eq. (3.13), we may impose the following global constraint,

$$\Phi_\rho \equiv \int_0^{2\pi} d\sigma d\rho(\partial_\tau X^i \partial_\rho X^i) = 0\,. \tag{3.16}$$

Thus, the light-cone toroidal membrane theory is described by the action (3.8) with the local constraint (3.11) and the global constraints (3.15) and (3.16).

Now we incorporate the local constraint (3.11) and the global constraints (3.15) and (3.16) as a gauge theory. Actually, the following gauge theory is equivalent to the system described by the action (3.8) with the constraints (3.11), (3.15) and (3.16),

$$\begin{aligned} S_g &= \frac{LT}{2}\int d\tau \int_0^{2\pi} d\sigma d\rho \left[(D_\tau X^i)^2 - \frac{1}{2L^2}\{X^i, X^j\}^2\right], \\ D_\tau X^i &= \partial_\tau X^i + u^\alpha \partial_\alpha X^i\,, \end{aligned} \tag{3.17}$$

[6]As for the closed string in the previous section, there is no counterpart of the local constraint.

where u^α is a gauge field satisfying the condition, $\partial_\alpha u^\alpha = 0$. The action (3.17) is invariant under the gauge transformations,

$$\begin{aligned}
\delta X^i &= -\epsilon^\alpha \partial_\alpha X^i \,, && (3.18)\\
\delta u^\alpha &= \partial_\tau \epsilon^\alpha + \partial_\beta \epsilon^\alpha u^\beta - \partial_\beta u^\alpha \epsilon^\beta \,, && (3.19)
\end{aligned}$$

where the parameter ϵ^α also satisfies the condition, $\partial_\alpha \epsilon^\alpha = 0$. The transformations generate (time-dependent) area-preserving diffeomorphisms. It is not difficult to see the equivalence between the gauge theory (3.17) and the mechanical system of the action (3.8) with the constraints, (3.11), (3.15) and (3.16). First, due to $\partial_\alpha u^\alpha = 0$, we can decompose the gauge field u^α into the following co-exact and harmonic parts [2, 3, 4],

$$u^\alpha = \frac{1}{L}\,\epsilon^{\alpha\beta}\,\partial_\beta A + \frac{1}{L}\,a^\alpha \,, \tag{3.20}$$

where the co-exact part $A = A(\tau, \sigma, \rho)$ is an arbitrary periodic function with respect to σ and ρ coordinates and the harmonic part $a^\alpha = a^\alpha(\tau)$ is independent of both σ and ρ coordinates. Similarly, due to $\partial_\alpha \epsilon^\alpha = 0$, we can decompose the parameter ϵ^α into the following co-exact and harmonic parts,

$$\epsilon^\alpha = \frac{1}{L}\,\epsilon^{\alpha\beta}\,\partial_\beta \Lambda + \frac{1}{L}\,\lambda^\alpha \,, \tag{3.21}$$

where the co-exact part $\Lambda = \Lambda(\tau, \sigma, \rho)$ is an arbitrary periodic function with respect to σ and ρ coordinates and the harmonic part $\lambda^\alpha = \lambda^\alpha(\tau)$ depends only on τ. The co-exact part Λ generates an invariant subgroup of the total group of the area-preserving diffeomorphisms [4]. Using the expressions (3.20) and (3.21), we can decompose the gauge transformations (3.18) and (3.19) as

$$\begin{aligned}
\delta X^i &= \frac{1}{L}\,\{\Lambda, X^i\} - \frac{1}{L}\,\lambda^\alpha \partial_\alpha X^i \,, && (3.22)\\
\delta A &= \partial_\tau \Lambda + \frac{1}{L}\,\{\Lambda, A\} + \frac{1}{L}\,\partial_\alpha \Lambda\, a^\alpha - \frac{1}{L}\,\lambda^\alpha \partial_\alpha A \,, && (3.23)\\
\delta a^\alpha &= \partial_\tau \lambda^\alpha \,. && (3.24)
\end{aligned}$$

Furthermore, by using the expression (3.20), the covariant derivative in eq. (3.17) becomes

$$D_\tau X^i = \partial_\tau X^i - \frac{1}{L}\{A, X^i\} + \frac{1}{L} a^\alpha \partial_\alpha X^i \,. \tag{3.25}$$

The Euler-Lagrange equations for X^i, A and a^α derived from the action (3.17), respectively, are given by

$$\begin{aligned}
D_\tau^2 X^i - \frac{1}{L^2}\{\{X^i, X^j\}, X^j\} &= 0 \,, && (3.26)\\
\{D_\tau X^i, X^i\} &= 0 \,, && (3.27)\\
\int_0^{2\pi} d\sigma d\rho\, (D_\tau X^i \partial_\sigma X^i) &= \int_0^{2\pi} d\sigma d\rho\, (D_\tau X^i \partial_\rho X^i) = 0 \,. && (3.28)
\end{aligned}$$

If we choose $A = a^\alpha = 0$ as gauge fixing conditions for the gauge transformations (3.22)–(3.24), eqs. (3.26), (3.27) and (3.28) agree with the Euler-Lagrange equation for X^i derived from the action (3.8), the local constraint (3.11) and the global constraints (3.15) and (3.16), respectively. Thus it has been shown that the gauge theory (3.17) is equivalent to the mechanical system described by the action (3.8) with the constraints (3.11), (3.15) and (3.16), i.e., the light-cone toroidal membrane theory.

A definitive difference between the closed string theory in the previous section and the toroidal membrane theory in this section is that it is easy to quantize the action (2.6) or (2.11) of the closed string, while it is not for the action (3.8) or (3.17) of the toroidal membrane due to non-linearity and non-renormalizability. As for the gauge theory, however, which is obtained by restricting the group of the area-preserving diffeomorphisms on the invariant subgroup generated by the parameter Λ, we can adopt the matrix regularization, so that we can quantize the gauge theory perturbatively. The action of such a gauge theory is as follows,[7]

$$S_{g'} = \frac{LT}{2}\int d\tau \int_0^{2\pi} d\sigma d\rho \left[\left(\partial_\tau X^i - \frac{1}{L}\{A, X^i\}\right)^2 - \frac{1}{2L^2}\{X^i, X^j\}^2\right], \tag{3.29}$$

and the gauge transformations are given by

$$\delta X^i = \frac{1}{L}\{\Lambda, X^i\}, \tag{3.30}$$

$$\delta A = \partial_\tau \Lambda + \frac{1}{L}\{\Lambda, A\}. \tag{3.31}$$

The matrix regularization is to regularize the infinite dimensional subgroup of the total group of the area-preserving diffeomorphisms by a finite dimensional group $\mathrm{U}(N)$ [5, 6]. In the matrix regularization, the infinite degrees of freedom due to the dependence of σ and ρ coordinates are mapped to $N \times N$ finite ones of hermitean matrices and the double integrals over σ and ρ and the Poisson bracket are mapped as follows,

$$\int_0^{2\pi} d\sigma d\rho \to \mathrm{Tr}, \tag{3.32}$$

$$\{\,\cdot\,,\,\cdot\,\} \to -i\,[\,\cdot\,,\,\cdot\,]. \tag{3.33}$$

We consider the matrix regularization of the gauge theory (3.17), which is invariant under the gauge transformation including the parameter λ^α. The simple maps, (3.32) and (3.33), however, cannot be applied straightforwardly because the covariant derivative (3.25) includes naked derivatives of ∂_σ and ∂_ρ which are not described by the Poisson brackets.

Let us fix the gauge of the action S_g in eq. (3.17). Henceforth we set $LT = 1$ for brevity. Following the standard procedure [11] we get the gauge fixed action,

$$\tilde{S}_g = S_g + \int d\tau \int_0^{2\pi} d\sigma d\rho \delta_B \left(-i\bar{c}^\alpha a^\alpha - i\bar{C}\left(\partial_\tau A + \frac{\xi}{2}B\right)\right)$$

[7] The action corresponds to that of a spherical membrane, which has no harmonic part from the outset.

$$
= \int d\tau \int_0^{2\pi} d\sigma d\rho \Bigg[\frac{1}{2}(D_\tau X^i)^2 - \frac{1}{4L^2}\{X^i, X^j\}^2 + i\bar{c}^\alpha \partial_\tau c^\alpha + b^\alpha a^\alpha - \tag{3.34}
$$

$$
-i\partial_\tau \bar{C} \left(\partial_\tau C + \frac{1}{L}\{C, A\} + \frac{1}{L}\partial_\alpha C a^\alpha - \frac{1}{L} c^\alpha \partial_\alpha A \right) + B\partial_\tau A + \frac{\xi}{2} B^2 \Bigg],
$$

where ξ is a gauge parameter, $C(\tau,\sigma,\rho)$, $\bar{C}(\tau,\sigma,\rho)$ and $B(\tau,\sigma,\rho)$ are the FP ghost, anti-ghost and NL B-field associated with the co-exact part of the gauge transformation, while the FP ghost $c^\alpha(\tau)$, anti-ghost $\bar{c}^\alpha(\tau)$ and NL B-field $b^\alpha(\tau)$ are associated with the harmonic part λ^α and they depend only on τ coordinate. $\tilde{S}_g$ is invariant under the BRS transformations,

$$
\begin{aligned}
\delta_B X^i &= \frac{1}{L}\{C, X^i\} - \frac{1}{L} c^\alpha \partial_\alpha X^i, \\
\delta_B A &= \partial_\tau C + \frac{1}{L}\{C, A\} + \frac{1}{L}\partial_\alpha C a^\alpha - \frac{1}{L} c^\alpha \partial_\alpha A, \\
\delta_B a^\alpha &= \partial_\tau c^\alpha, \\
\delta_B C &= \frac{1}{2L}\{C, C\} - \frac{1}{L} c^\alpha \partial_\alpha C, \\
\delta_B c^\alpha &= 0, \\
\delta_B \bar{C} &= iB, \\
\delta_B \bar{c}^\alpha &= ib^\alpha, \\
\delta_B B &= \delta_B b^\alpha = 0.
\end{aligned} \tag{3.35}
$$

The BRS charge is given by

$$
Q_B = -\frac{1}{L}\int_0^{2\pi} d\sigma d\rho \Big[\{D_\tau X^i, X^i\}\, C + D_\tau X^i \partial_\alpha X^i\, c^\alpha -
- \frac{i}{2}\{\partial_\tau \bar{C}, C\}\, C - i\partial_\tau \bar{C}\, \partial_\alpha C\, c^\alpha \Big]. \tag{3.36}
$$

We concentrate on the contribution of the BRS quartet $\{a^\alpha, b^\alpha, c^\alpha, \bar{c}^\alpha\}$ associated with the global constraints, which we call GC-quartet. We extract the terms which include the member(s) of the GC-quartet from $\tilde{S}_g$ (3.34),

$$
\begin{aligned}
S_{GCq} &= \int d\tau \int_0^{2\pi} d\sigma d\rho \Bigg[\left(\partial_\tau X^i - \frac{1}{L}\{A, X^i\} \right) \frac{1}{L} a^\alpha \partial_\alpha X^i + \frac{1}{2L^2}\left(a^\alpha \partial_\alpha X^i\right)^2 + \\
&\qquad + i\bar{c}^\alpha \partial_\tau c^\alpha + b^\alpha a^\alpha - i\frac{1}{L}\partial_\tau \bar{C}\left(\partial_\alpha C\, a^\alpha - c^\alpha \partial_\alpha A\right) \Bigg] \\
&= \int d\tau \int_0^{2\pi} d\sigma d\rho \left[\tilde{b}^\alpha a^\alpha + i\bar{c}^\alpha \partial_\tau c^\alpha + i\frac{1}{L}\partial_\tau \bar{C}\left(c^\alpha \partial_\alpha A\right) \right],
\end{aligned} \tag{3.37}
$$

where

$$
\tilde{b}^\alpha = b^\alpha + \int_0^{2\pi} \frac{d\sigma d\rho}{(2\pi)^2} \frac{1}{L} \left[\left(\partial_\tau X^i - \frac{1}{L}\{A, X^i\} \right) \partial_\alpha X^i + \frac{1}{2L}\partial_\alpha X^i a^\beta \partial_\beta X^i - i\partial_\tau \bar{C} \partial_\alpha C \right]. \tag{3.38}
$$

The action S_{GCq} (3.37) consists of free part (the first two terms) and the interaction part (the last term). However, we should notice that the interaction term never contributes to any correlation functions of $X^i, A, C, \bar{C}$ and B *since there is no interaction term which includes both $\bar{c}^\alpha$ and C in the total action $\tilde{S}_g$*.[8] This means that S_{GCq} is essentially free and hence it contributes an overall trivial factor to the correlation functions of $X^i, A, C, \bar{C}$ and B in the path-integral formalism. Thus, $\tilde{S}_g$ (3.34) is equivalent to the gauge fixed action for the gauge theory $S_{g'}$ (3.29) up to the essentially free action for the GC-quartet. Note that after (path-)integrating out the GC-quartet we can matrix-regularize the action, i.e., $\tilde{S}_g - S_{GCq}$, by the simple maps (3.32) and (3.33). We should also notice that the above mentioned facts always hold if we adopt $\delta_B(-i\bar{c}^\alpha a^\alpha)$ to fix the λ^α gauge transformation and do not use the GC-quartet in the gauge fixing function F_Λ of the Λ gauge transformation, i.e., we adopt $\delta_B(-i\bar{c}^\alpha a^\alpha - i\bar{C}F_\Lambda)$ where F_Λ is an arbitrary function which does not depend on any of $\{a^\alpha, b^\alpha, c^\alpha, \bar{c}^\alpha\}$.[9] Hence, as to the Λ gauge transformation, we can adopt not only $F_\Lambda = \partial_\tau A + \xi B/2$ but also such a gauge as was used in the wrapped supermembrane theory [9, 10].

Finally we comment on the quantum mechanical study of the double-dimensional reduction of the eleven dimensional supermembrane [9, 10]. In ref. [9], the problem was first analyzed in the world-volume field theory of the wrapped supermembrane in the path-integral formalism. Then in ref. [10], the similar analysis was performed in the world-volume field theory in the matrix-regularized form of the wrapped supermembrane, i.e., matrix string theory [14, 15]. In those analyses, however, the global constraints, which should be the extended version of (3.15) and (3.16) to the wrapped supermembrane theory, were not taken into account. As for ref. [10] in particular, it is because the matrix-regularized forms of the global constraints were not obvious. Actually, in the standard derivation [14, 15] of matrix string theory based on Seiberg and Sen's arguments [16, 17] and the compactification prescription by Taylor [18], such matrix-regularized global constraints do not appear naturally. The difference between $\tilde{S}_g$ (3.34) and the gauge fixed version of $S_{g'}$ (3.29) is only the essentially free action for the GC-quartet, $\{a^\alpha, b^\alpha, c^\alpha, \bar{c}^\alpha\}$, which do not affect the quantum mechanical study of the double-dimensional reduction in the path-integral formalism. This reminds us of QED in the path-integral formalism, where the FP ghosts can be free and contribute only to the vacuum energy. In fact, the determinant of the FP ghosts cancel out the contribution from the longitudinal and scalar modes of the gauge field. However such a determinant of the FP ghosts is not important to physical amplitudes. Thus, our results of this paper justify the analyses of refs. [9] and [10], where the global constraints were not taken into account.

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[8]In the path-integral language, this is due to the fact that the FP determinant of the triangular matrix is equal to that of the diagonal matrix.

[9]We can adopt such a weaker condition for F_Λ as $\partial F_\Lambda / \partial \bar{c}^\alpha = 0$.

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